INTEGRAL P-ADIC HODGE THEORY, TALK 2 (PERFECTOID RINGS, $\mathbf{A}_{\inf}$ AND THE PRO-ÉTALE SITE)

POL VAN HOFTEN (NOTES BY JAMES NEWTON)

1. Witt vectors, $\mathbf{A}_{\inf}$ and integral perfectoid rings

The first part of the talk will cover Witt vectors, $\mathbf{A}_{\inf}$ and integral perfectoid rings, following section 3 of [1].

1.1. **Setting.** We fix a prime p, and consider rings S which are π-adically complete and separated with respect to an element $\pi|p$. We define the *tilt*

$$S^\flat = \varprojlim_{x\mapsto x^p} S/pS$$

and remark that

$$S^b = \varprojlim_{x\mapsto x^p} S$$

as monoids (the usual proof works).

Examples.

- $S = \mathbb{Z}_p$, $S^\flat = \mathbb{F}_p$
- If $S = \widehat{\mathbb{Z}_p[T]}$ (the p-adic completion) then $S/pS = \mathbb{F}_p[T]$ and $S^\flat = (\mathbb{F}_p[T])^{\mathrm{perf}} = \mathbb{F}_p$.
- If $S = \mathbb{Z}_p^{\mathrm{cyc}} = \widehat{\mathbb{Z}_p[\mu_{p^\infty}]}$ then $S/pS = \mathbb{F}_p[T^{1/p^\infty}]/T$ and $S^\flat = \mathbb{F}_p[[T^{1/p^\infty}]]$.

Definition 1.2. $\mathbf{A}_{\inf}(S) = W(S^\flat)$.

Applying the definition to the above examples, we get (respectively):

- $\mathbf{A}_{\inf}(S) = W(\mathbb{F}_p)$
- $\mathbf{A}_{\inf}(S) = W(\mathbb{F}_p)$
- $\mathbf{A}_{\inf}(S) = \mathbb{Z}_p\widehat{[[T^{1/p^\infty}]]}$.

We have a map $\theta : \mathbf{A}_{\inf}(S) \to S$ which lifts the map $S^\flat \to S/pS$ given by projecting to the bottom component of the inverse limit. We can define θ as

$$\theta\left(\sum_{i\geq 0}[a_i]p^i\right) = \sum_{i\geq 0} a_i^\sharp p^i$$

where $()^\sharp$ is the multiplicative map $S^\flat \to S$ given by considering $S^\flat = \varprojlim_{x\mapsto x^p} S$ and projecting onto the first component.

When S is perfectoid (definition is to come), the map θ will be surjective (this will follow from surjectivity of $S^\flat \to S/pS$).

Next we are going to define maps $\theta_r : \mathbf{A}_{\inf}(S) \to W_r(S)$ (with $\theta_1 = \theta$). First we have some recollections on Witt vectors.

1.3. **Witt vectors.** For a ring A, we can define the (p-typical) Witt vectors $W(A)$. As sets we have $W(A) = A^{\mathbb{N}}$. The Witt vectors come equipped with ring homomorphisms (the *ghost components*) for $r \geq 0$:

$$W(A) \xrightarrow{\omega_r} A$$
$$(a_0, \ldots, a_r, \ldots) \mapsto a_0^{p^r} + pa_1^{p^{r-1}} + \cdots + p^r a_r$$

Truncating to the first r entries $(a_0, \ldots, a_{r-1})$, we get the truncated Witt ring $W_r(A)$. We have obvious restriction maps $R : W_{r+1}(A) \to W_r(A)$ and also a Witt vector Frobenius $F : W_{r+1}(A) \to W_r(A)$. When A has characteristic p, F is simply given by $F(a_0, \ldots, a_r) = (a_0^p, \ldots, a_{r-1}^p)$, so we can write $R\phi = \phi R = F$, where ϕ is the map raising each component to the pth power. On ghost components, we have $\omega_i(Fx) = \omega_{i+1}(x)$ for all $x \in W_{r+1}$ and $i \leq r - 1$.

Remark 1.4. We have $\mathbf{A}_{\mathrm{inf}}(S) = \varprojlim_R W_r(S^\flat)$. Since $R^i\phi^i = F^i$ and $S^\flat$ is perfect, the maps

$$\phi^r : W_r(S^\flat) \to W_r(S^\flat)$$

induce an isomorphism

$$\phi^\infty : \varprojlim_F W_r(S^\flat) \xrightarrow{\sim} \varprojlim_R W_r(S^\flat).$$

Taking Witt vectors commutes with inverse limits so we have

$$\varprojlim_F W_r(S^\flat) = \varprojlim_F \varprojlim_\phi W_r(S/pS) = \varprojlim_\phi \varprojlim_F W_r(S/pS).$$

Finally, ϕ is an automorphism of $\varprojlim_F W_r(S/pS)$ (use $\phi R = R\phi = F$) so we conclude that $\mathbf{A}_{\mathrm{inf}}(S) = \varprojlim_F W_r(S/pS)$.

Remark 1.5. The multiplicative map $()^\sharp : S^\flat \to S$ extends to a multiplicative bijection $S^\flat \to \varprojlim_{x \mapsto x^p} S$ (inverse to the canonical multiplicative map induced by $S \to S/pS$). At the level of Witt vectors, the canonical ring homomorphism $\varprojlim_F W_r(S) \to \varprojlim_F W_r(S/pS)$ is in fact an isomorphism. So we have

$$\mathbf{A}_{\mathrm{inf}}(S) = \varprojlim_F W_r(S).$$

Projecting to the rth component in the inverse limit, we obtain a ring homomorphism

$$\widetilde{\theta}_r : \mathbf{A}_{\mathrm{inf}}(S) \to W_r(S).$$

Definition 1.6. We set $\theta_r = \widetilde{\theta}_r \circ \phi^r$. Here $\phi = W(\phi)$, the map induced by functoriality from the p-power map on $S^\flat$.

Lemma 1.7. *We have* $\widetilde{\theta}_r([x]) = [x^{(r)}]$ *for* $x \in S^\flat$ *where we write* $x = (x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x \mapsto x^p} S$. *We therefore have* $\theta_r([x]) = [x^{(0)}] = [x^\sharp]$ *and in particular* $\theta_1 = \theta$.

1.8. **Integral perfectoid rings.**

Definition 1.9. S is (integral) perfectoid if S is π-adically complete and:
1. $\pi^p | p$
2. The Frobenius $\phi : S/pS \to S/pS$ is surjective
3. $\theta : \mathbf{A}_{\mathrm{inf}}(S) \to S$ has principal kernel

Examples.

- $\mathbb{Z}_p^{\mathrm{cyc}}$
- $\mathbb{Z}_p^{\mathrm{cyc}}\langle T^{1/p^\infty}\rangle$
- The p-adic completion of $\mathbb{Z}_p[[X]] \otimes \mathbb{Z}_p[p^{1/p^\infty}, X^{1/p^\infty}]$
- If A is an integral domain with $p \notin A^\times$, the p-adic completion of an absolute integral closure of A is a perfectoid ring.

Lemma 1.10. *The following are equivalent:*

(1) $\phi : S/pS \to S/pS$ *is surjective*
(2) $F : W_{r+1}(S) \to W_r(S)$ *is surjective for all* r
(3) $\theta_r : \mathbf{A}_{\mathrm{inf}}(S) \to W_r(S)$ *is surjective for all* r

Remark 1.11. If $F : W_2(S) \to S$ is surjective, then part (1) in the above holds. Use the fact that $F(a_0, a_1) = a_0^p + pa_1$.

Lemma 1.12. *If* π *is not a zero divisor,* ϕ *is surjective and* $\ker(\theta)$ *is principal then*

$$\phi : S/\pi S \to S/\pi^p S$$

is an isomorphism.

Now we consider $S = \mathcal{O}_K$, where K is a perfectoid field (the old definition) with $\mu_{p^\infty} \subset K$ and a fixed compatible system of p-power roots of unity (ζ_{p^i}). Then we define $\epsilon \in S^\flat$ by $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$ and so $[\epsilon] \in \mathbf{A}_{\mathrm{inf}}(S)$.

Fact: $\ker(\theta)$ is generated by $1 + [\epsilon^{1/p}] + \cdots + [\epsilon^{1/p}]^{p-1}$. The map

$$\theta_\infty := \varprojlim_r \theta_r : \mathbf{A}_{\mathrm{inf}} \to W(S)$$

has kernel generated by $\mu = [\epsilon] - 1$, and if K is spherically complete θ_∞ is surjective.

To compare with the more standard perfectoid terminology: if S is perfectoid and flat over $\mathbb{Z}_p$ then $(S[\frac{1}{\pi}], S)$ is a perfectoid Huber pair — the ideal defining the topology on S is (π).

2. The pro-étale site of an adic space

All our adic spaces will be locally Noetherian adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, so in particular they will be analytic.

Definition 2.1. A map $f : X \to Y$ of adic spaces is finite étale if it is affinoid and for an open cover of Y by affinoids $\mathrm{Spa}(A, A^+)$ the pull back $\mathrm{Spa}(B, B+)$ of X is finite étale over $\mathrm{Spa}(A, A^+)$ — i.e. $A \to B$ is finite étale and B^+ is the integral closure of A^+ in B.

Definition 2.2. A map of adic spaces $f : X \to Y$ is étale if $\forall x \in X$ there is an open neighbourhood $x \in U \subset X$ and $f(U) \subset V$ open in Y such that f factors as

$$\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & W \\ & \searrow^{f} & \downarrow g \\ & & V \end{array} \qquad (2.2.0)$$

where i is an open immersion and g is finite étale.

Remark 2.3. This definition does not work in algebraic geometry! (Note by Pol: I am now unsure of this)

We obtain sites $X_{f\text{ét}} \subset X_{\text{ét}}$ with coverings jointly surjective families of (finite) étale maps.

Now we want to introduce the pro-étale site. Two advantages of this will be: firstly inverse limits of sheaves will behave well, so we can compute l-adic cohomology groups as genuine cohomology groups of a sheaf $\underline{\mathbb{Z}_l} = \varprojlim_n \underline{\mathbb{Z}/l^n\mathbb{Z}}$, rather than inverse limits of cohomology groups. Secondly, every adic space will be pro-étale-locally perfectoid.

The pro-étale site is sandwiched between the pro-categories given by towers of finite étale, or étale, maps:

$\text{pro} - X_{f\text{ét}} \subset X_{\text{proét}} \subset \text{pro} - X_{\text{ét}}$.

We don't want to take the whole of $\text{pro} - X_{\text{ét}}$ because it includes maps which are not open (for example, think of a tower of discs of shrinking radii).

2.4. **Pro-categories and the pro-étale site.** Our index categories I are co-filtered: this meams we have the following two properties:

- For every pair of objects i, j of I we have another object k with morphisms $k \to i$ and $k \to j$.
- For every pair of morphisms $f, g : i \to j$ in I we have an object k and a morphism $h : k \to i$ such that $fh = gh$.

Now given a category C, the objects of the pro-category $\text{pro}-C$ are functors $I \to C$ from (small) co-filtered categories I. Given $U \in \text{pro} - X_{\text{ét}}$ we write $U = \text{“}\varprojlim_i\text{”} U_i$ (i varies over objects of the index category I).

Definition 2.5. We define a full subcategory $X_{\text{proét}} \subset \text{pro} - X_{\text{ét}}$ by saying that U is in $X_{\text{proét}}$ if U is isomorphic to $\text{“}\varprojlim_i\text{”} U_i$ with

(1) $U_i \to X$ étale for all i (this is automatic)
(2) $U_i \to U_j$ finite étale and surjective for all $i \to j$.

Remark 2.6. We can modify the second condition in the above definition by substituting ‘for all’ with ‘for a cofinal system of’, and get an equivalent definition.

Definition 2.7. Given $U = \text{“}\varprojlim_i\text{”} U_i$ in $X_{\text{proét}}$ we define the topological space $|U| = \varprojlim_{i \in I} |U_i|$.

Definition 2.8. (1) Given a morphism $U \to V$ in $X_{\text{proét}}$ we say that $U \to V$ is étale (resp. finite étale) if there exists $U_0 \to V_0$ an étale (resp. finite étale) map of adic spaces and $V \to \underline{V}_0$ (the constant pro-object given by V_0) such that $U \cong \underline{U}_0 \times_{\underline{V}_0} V$.
(2) We say that $U \to V$ is pro-étale if $U \cong \varprojlim_k A_k$ (here $A_k \in X_{\text{proét}}$) with $A_k \to V$ étale and $A_k \to A_{k'}$ finite étale surjective.
(3) Finally, we define coverings in $X_{\text{proét}}$ to be $\{U_i \to V\}$ such that the $|U_i|$ cover V, each map $U_i \to V$ is pro-étale, plus a set-theoretic condition (which is automatic if the inverse limits are over countable index sets).

Proposition 2.9.
- *This definition of covering makes $X_{\text{pro}\acute{e}t}$ into a site — in particular, pro-étale maps are preserved under composition and base change.*
- *Pro-étale maps are open (i.e. $|f| : |U| \to |V|$ is open)*
- *Given $V \subset |U|$ open, we can find $W \to U$ in $X_{\text{pro}\acute{e}t}$ with $|W| \cong V$.*
- *There is a map of sites*

$$\nu : X_{\text{pro}\acute{e}t} \to X_{\acute{e}t}$$

(induced by the functor in the other direction, taking something étale over X to the associated constant pro-object). In particular we have functors ν_*, ν^* on the categories of abelian sheaves.

- *If $\mathcal{F} \in \mathrm{Ab}(X_{ét})$ is a sheaf of abelian groups (and X is qcqs) we have*
$$H^i(X_{ét}, \mathcal{F}) = H^i(X_{\text{pro}ét}, \nu^*\mathcal{F}).$$
- *If $\mathcal{F} \in \mathrm{Ab}(X_{ét})$ the natural adjunction map $\mathcal{F} \to R\nu_*\nu^*\mathcal{F}$ is an isomorphism.*

Remark 2.10. If $U =$ "$\varprojlim_i$" $\mathrm{Spa}(A_i, A_i^+)$ then $\nu^*\mathcal{F}(U) = \varinjlim_i \mathcal{F}(\mathrm{Spa}(A_i, A_i^+))$.

2.11. **Sheaves on the pro-étale site.** Here are a bunch of sheaves on $X_{\text{proét}}$:

- $\mathcal{O}_X^+$ (we have $\mathcal{O}_X^+($"$\varprojlim_i$" $\mathrm{Spa}(A_i, A_i^+)) = \varinjlim_i A_i^+$)
- $\mathcal{O}_X$, $\widehat{\mathcal{O}}_X^+ = \varprojlim \mathcal{O}_X^+/p^n$, $\widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X^+[\frac{1}{p}]$
- $\widehat{\mathcal{O}}_{X^\flat}^+ = \varprojlim_\phi \mathcal{O}_X^+/p$
- $\mathbf{A}_{\inf,X} = W(\widehat{\mathcal{O}}_{X^\flat}^+)$

Remark 2.12. In general the sections of these sheaves are not easy to compute: $\widehat{\mathcal{O}}_X^+(U)$ may not equal the p-adic completion of $\varinjlim_i A_i^+$. But they do behave well on affinoid perfectoids.

Definition 2.13. $U =$ "$\varprojlim_i$" $\mathrm{Spa}(A_i, A_i^+)$ is affinoid perfectoid if
$$(\widehat{\varinjlim_i A_i^+}[\frac{1}{p}], \widehat{\varinjlim_i A_i^+})$$
is a perfectoid Huber pair.

Example 2.13.1. Let $X_n = \mathrm{Spa}(K\langle T^{\pm 1/p^n}\rangle, \mathcal{O}_K\langle T^{\pm 1/p^n}\rangle)$ for K a perfectoid field. Set $X = X_1$. Then "$\varprojlim_n$"X_n is an affinoid perfectoid in $X_{\text{proét}}$.

References

[1] Bhatt, B., Morrow, M. and Scholze, P. *Integral p-adic Hodge theory*, arXiv:1602.03148.